Assignment 2 covering material from Lectures 1-27.
Due by 11:59pm on 5 June.

Instructions: Please write detailed solutions to the following problems. Quality of exposition will comprise a significant number of marks. Clarity, neatness, and style count.

1. (a) Show that if $|X| \le a$ and $\mathbb{E}[X] = 0$, then for any $\theta \in \mathbb{R}$,

$$\mathbb{E}[e^{\theta X}] \le e^{\theta^2 a^2/2}.$$

Hints. (1) The convexity of $x \mapsto e^{\theta x}$ implies that the line going through $(-a, e^{-\theta a})$ and $(a, e^{\theta a})$ lies above $e^{\theta x}$ for $x \in [-a, a]$. (2) It may help to consider the power series of cosh.

(b) Let $X_1, X_2, \dots, X_j$ be independent mean-zero random variables with the property that there are $a_i > 0$, $i = 1, \dots, j$, such that, for $\theta \in \mathbb{R}$,

$$\mathbb{E}[e^{\theta X_i}] \le e^{\theta^2 a_i^2/2}.$$

Show that for $S_j = \sum_{i=1}^j X_i$ and any $t \ge 0$,

$$\mathbb{P}\big(|S_j| \ge t\big) \le 2e^{-t^2/(2\sum_{i=1}^j a_i^2)}.$$

2. Let W be the total population size of a branching process with offspring distribution $\mathscr{L}(X)$ with $\lambda := \mathbb{E}[X] \ne 1$ and $\zeta = \mathbb{P}(W = \infty)$. Let $X_1, X_2, \dots$ be i.i.d. with distribution $\mathscr{L}(X)$ and write $S_j = \sum_{i=1}^j X_i$. Assume that there are positive constants c_1, c_2, such that for any $t \ge 0$,

$$\mathbb{P}\big(|S_j - j\lambda| \ge t\big) \le c_1 e^{-c_2 t^2/j}. \tag{1}$$

(a) Show that for any $k \in \mathbb{N}$ with $k \ge 1/(1-\lambda)$,

$$\mathbb{P}\big(W \ge k\big) \le \zeta + c_1 e^{-c_2(k(1-\lambda)^2 - 2(1-\lambda))}\big(1 - e^{-c_2(1-\lambda)^2}\big)^{-1}.$$

(b) Show that if $\lambda > 1$, then

$$\mathbb{P}(W = \infty | W \ge k) \ge 1 - \zeta^{-1} c_1 e^{-c_2(k(1-\lambda)^2 - 2(1-\lambda))}\big(1 - e^{-c_2(1-\lambda)^2}\big)^{-1}.$$

Note that the result above says that if (1) *holds, then a supercritical branching process either dies out "quickly" (with approximate probability* $1 - \zeta$*), or lives forever (with probability* ζ*). This is the same property we used to show that a supercritical Erdős-Rényi random graph has a unique giant component.*

3. Fix $n \in \mathbb{N}$ and $p \in (0, 1)$ and let W be the number of isolated edges in $G \sim \mathrm{ER}(n, p)$.

(a) Compute $\mu := \mathbb{E}[W]$.

(b) For each $\{i, j\} \subseteq \{1, \dots, n\}$, let X_{ij} be the indicator that edge $\{i, j\}$ is isolated in G. Construct couplings of $\mathscr{L}\big((X_{ij})_{i \ne j}\big)$ and $\mathscr{L}\big((X_{ij})_{i \ne j} | X_{k\ell} = 1\big)$ for each $\{k, \ell\} \subseteq \{1, \dots, n\}$, and use Theorem 25.3 from Lecture 25 to show that

$$d_{\mathrm{TV}}\big(\mathscr{L}(W), \mathrm{Po}(\mu)\big) \le \min\big\{\mu, \mu^2\big\}\left(\frac{4}{n} + \frac{1 - (1-p)^4}{(1-p)^4}\right).$$