Lecture 22: Component sizes in supercritical Erdős-Rényi random graphs

The last loose end we have is to prove Lemma 20.3 from Lecture 20. For $\lambda > 1$, let ζ_λ denote the probability that a branching process with Poisson(λ) offspring distribution does not go extinct. Recall from Example 12.3 of Lecture 12 that ζ_λ satisfies

$$\zeta_\lambda = 1 - e^{-\lambda\zeta_\lambda}. \tag{22.1}$$

Lemma 20.3. *For $v \in \{1, \dots, n\}$, let C_v denote the size of the component containing vertex v in $G \sim \mathrm{ER}(n, \lambda/n)$ with $\lambda > 1$, let $0 < \gamma < \zeta_\lambda$, and let $V_k = \sum_{v=1}^n \mathbb{I}[k < C_v \le \gamma n]$. Then for any $\alpha \in (0, 1/2)$ and $\beta \in (0, 1-2\alpha)$, and constant $c > 0$, there is a $\kappa > 0$ such that for $k = \lceil (1+\kappa) I_\lambda^{-1} \log(n) \rceil$,*

$$\mathbb{P}(V_k > c\, n^{1-\alpha}) = \mathrm{O}(n^{-\beta}).$$

The proof follows the usual first moment route, using Chebychev's inequality and then showing that $\mathbb{E}[V_k]/n^{1-\alpha} = \mathrm{O}(n^{-\beta})$. Since $\mathbb{E}[V_k] = n\mathbb{P}(k < C_v \le \gamma n)$, we again approximate the probability on the right hand side. But at the top of the interval, we are in a region where the Poisson branching process approximation of the Erdős-Rényi exploration process is no longer good, so we need another approximation result.

Lemma 22.2. *For $v \in \{1, \dots, n\}$, let C_v denote the size of the component containing vertex v in $G \sim \mathrm{ER}(n, \lambda/n)$ and for $t = 0, 1, \dots,$ let $X_t \sim \mathrm{Bi}\big(n-1, 1 - \big(1 - \frac{\lambda}{n}\big)^t\big)$. Then*

$$\mathbb{P}(C_v = t) \le \mathbb{P}(X_t = t - 1). \tag{22.2}$$

Before proving the lemma a couple of remarks.

Remark 22.3. To understand how this lemma will help us to show Lemma 20.3, note that

$$\mathbb{E}[X_t] \approx n\Big(1 - \big(1 - \tfrac{\lambda}{n}\big)^t\Big) \approx n\big(1 - e^{-\lambda t/n}\big). \tag{22.3}$$

From the binomial tail bound exercise associated to Lecture 16, we know for any $0 < \delta < 1$,

$$\mathbb{P}\big(X_t \le (1-\delta)\mathbb{E}[X_t]\big) \le \exp\big\{-\mathbb{E}[X_t]\delta^2/2\big\}. \tag{22.4}$$

The right hand side of (22.2) is upper bounded by the tail $\mathbb{P}(X_t \le t)$, which, by (22.4), will be small (noting that $\mathbb{E}[X_t]$ is of order t) if there is a $0 < \delta < 1$ not depending on t such that

$$\frac{t}{\mathbb{E}[X_t]} < 1 - \delta.$$

Using (22.3), this inequality is roughly the same as

$$\frac{t/n}{1 - e^{-\lambda(t/n)}} < 1 - \delta, \tag{22.5}$$

and, using the definition of ζ_λ given at (22.1) and properties of the function $z \mapsto z - 1 + e^{-\lambda z}$, (22.5) will be satisfied when there is $\gamma \in (0, \zeta_\lambda)$ such that $t < n\gamma$. Thus we can bound each term in the sum

$$\mathbb{E}[V_k] = n \sum_{t \in (k, \gamma n]} \mathbb{P}(C_v = t) \tag{22.6}$$

using (22.2), and the bound on the sum will be geometric with base a negative power of n that can be made as small as necessary by increasing κ. (The proof will make all of this precise.)

Remark 22.4. Comparing to the earlier branching process bound/approximation, for $t/n \ll 1$, we have

$$1 - \big(1 - \tfrac{\lambda}{n}\big)^t \approx \lambda t/n, \tag{22.7}$$

and so, by the approximation of the Binomial distribution by the Poisson,

$$X_t \overset{d}{\approx} Y_{\lambda t},$$

where $Y_{\lambda t} \sim \mathrm{Po}(\lambda t)$. This with Lemma 22.2 implies that $\mathbb{P}(C_v = t) \approx \mathbb{P}(Y_{\lambda t} = t - 1)$, which compares well to the previous approximation

$$\mathbb{P}(C_v \geq k) \leq \sum_{j \geq k} \mathbb{P}(Y_{\lambda j} = j - 1).$$

The advantage of Lemma 22.2 is that it doesn't rely on (22.7) and, in particular, gives good bounds all through the range $t < \zeta_\lambda n$.

Proof of Lemma 22.2. In order for $C_v = t$, we need to select the $t - 1$ other vertices in the component with v, which can be done in $\binom{n-1}{t-1}$ ways, then these t vertices must not connect to the remaining $n - t$ vertices, which happens with probability $(1 - \lambda/n)^{t(n-t)}$, and then the t vertices chosen for the component must have enough edges to be connected. Thus, writing $p = \lambda/n$, if $G(t, p) \sim \mathrm{ER}(t, p)$, then

$$\mathbb{P}(C_v = t) = \binom{n-1}{t-1}(1-p)^{t(n-t)}\mathbb{P}\big(G(t,p) \text{ is connected}\big). \tag{22.8}$$

Now, to compute $\mathbb{P}\big(G(t,p) \text{ is connected}\big)$, recall the Erdős-Rényi exploration process started from some fixed vertex v, described first in Lecture 11 and the notation there. In particular, the neutral vertex counting process $(N_s)_{s \geq 0}$ satisfies $N_0 = t - 1$ and, on the event that $N_{s-1} \geq t - s$,

$$\mathscr{L}\big(N_s | N_{s-1}\big) = \mathrm{Bi}(N_{s-1}, 1 - p). \tag{22.9}$$

This is because given the process up to time $s - 1$, if $N_{s-1} \geq t - s$ then there are still active vertices, and then at step s each of the N_{s-1} neutral vertices are made active independently with probability p, which is the same as each staying neutral with probability $1 - p$. Once $N_{s-1} < t - s$, the process stops and (22.9) is no longer satisfied. However, we define a new process $(N'_s)_{s \geq 0}$ satisfying

$$\mathscr{L}\big(N'_s | N'_{s-1}\big) = \mathrm{Bi}(N'_{s-1}, 1 - p) \tag{22.10}$$

for all $s \geq 1$, and coupled to $(N_s)_{s \geq 0}$ in the obvious way so that $N'_s \leq N_s$ for $s \geq 0$. Noting that the only way $G(t, p)$ is connected is if the exploration process of every (any) fixed vertex ends with no neutral vertices, we have

$$\big\{G(t,p) \text{ is connected}\big\} = \{N_t = 0\} \subseteq \{N'_t = 0\}.$$

The lemma now follows from this after we show that for any $0 \leq s \leq t$,

$$N'_s \sim \mathrm{Bi}(t - 1, (1 - p)^s),$$

which we show by induction. For $s = 0$, the result follows since $N'_0 = t - 1$. For $s = 1$, the result follows from (22.10). The induction step follows from the following fact: If $M \sim \mathrm{Bi}(m, q)$ and $\mathscr{L}(M' | M = \ell) = \mathrm{Bi}(\ell, q')$, then $M' \sim \mathrm{Bi}(m, qq')$. To show this fact, compute the generating function of M' as follows.

$$\mathbb{E}[z^{M'}] = \mathbb{E}\big\{\mathbb{E}[z^{M'} | M]\big\} = \mathbb{E}\big[(1 - q' + q'z)^M\big] = \big(1 - q + q(1 - q' + q'z)\big)^m = (1 - qq' + qq'z)^m.$$

□

We can now prove Lemma 20.3 as outlined in Remark 22.3.

Proof of Lemma 20.3. As described after the statement of the lemma, we use Chebychev's inequality, (22.6), and Lemma 22.2 to find

$$\begin{aligned}\mathbb{P}(V_k > c\,n^{1-\alpha}) &\le \frac{\mathbb{E}[V_k]}{c\,n^{1-\alpha}} \\ &= c^{-1}n^{\alpha}\sum_{t\in(k,\gamma n]}\mathbb{P}(C_v = t) \\ &\le c^{-1}n^{\alpha}\sum_{t\in(k,\gamma n]}\mathbb{P}(X_t \le t),\end{aligned} \tag{22.11}$$

where $X_t \sim \mathrm{Bi}\big(n-1, 1-\big(1-\frac{\lambda}{n}\big)^t\big)$. Now note, ,

$$\begin{aligned}\mathbb{E}[X_t] &= (n-1)\Big(1-\Big(1-\frac{\lambda}{n}\Big)^t\Big) \\ &\ge (n-1)\big(1-e^{-\lambda t/n}\big) \\ &= t\Big(\frac{n-1}{n}\Big)\Big(\frac{1-e^{-\lambda t/n}}{t/n}\Big) \\ &\ge t\Big(\frac{n-1}{n}\Big)\Big(\frac{1-e^{-\lambda\gamma}}{\gamma}\Big),\end{aligned} \tag{22.12}$$

where the last inequality is because $z \mapsto (1-e^{-\lambda z})/z$ is a decreasing function for $z > 0$ and $t/n \le \gamma$. Now, since the for $\lambda > 1$, the function $z \mapsto z - 1 + e^{-\lambda z}$ defined on $z \ge 0$ has one critical point, a strictly positive second derivative, and equals zero at $z = 0$ and $z = \zeta_\lambda$, we have that for $0 < z < \zeta_\lambda$,

$$z - 1 + e^{-\lambda z} < 0.$$

and therefore

$$1 - e^{-\lambda\gamma} > \gamma \implies \frac{1-e^{-\lambda\gamma}}{\gamma} > 1.$$

Plugging this into (22.12) gives, for large enough n and a $u > 1$, only depending on γ,

$$\mathbb{E}[X_t] \ge tu,$$

and so, using (22.4) with $\delta = 1 - u^{-1} \in (0,1)$,

$$\mathbb{P}(X_t \le t) \le \mathbb{P}\big(X_t \le \mathbb{E}[X_t]u^{-1}\big) \le \exp\Big\{-\mathbb{E}[X_t]\big(1-u^{-1}\big)^2/2\Big\} \le \exp\Big\{-tu\big(1-u^{-1}\big)^2/2\Big\}.$$

Putting this inequality into (22.11), we have for $k = \lceil (1+\kappa)I_\lambda^{-1}\log(n)\rceil$

$$\begin{aligned}\mathbb{P}(V_k > c\,n^{1-\alpha}) &\le c^{-1}n^{\alpha}\sum_{t\in(k,\gamma n]}\exp\Big\{-tu\big(1-u^{-1}\big)^2/2\Big\} \\ &= \mathrm{O}\Big(n^{\alpha}\exp\big\{-\log(n)(1+\kappa)I_\lambda^{-1}u\big(1-u^{-1}\big)^2/2\big\}\Big),\end{aligned}$$

and the right hand side can be made smaller than any negative power of n by increasing κ. □